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ON THE DENSITY OF HYPONORMAL OPERATORS

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ABSTRACT

The set of hyponormal operators whose squares are not hyponormal is norm dense in the set of all hyponormal operators.

A well known example of Halmos [1] exhibits a hyponormal operator T whose square is not hyponormal. The first author has shown that the class Q of such operators is strongly dense in the set H_0 of all hyponormal operators. The purpose of this note is to show Q is actually norm dense in H_0.

We begin with the following

LEMMA. *Let S be the unilateral shift; thus $Se_n = e_{n+1}$ for $n = 1, 2, \cdots$ where $\{e_j\}_1^\infty$ is an orthonormal basis for $\mathcal{H}$.*

Set $T_{a,b} = I + aS + bS^*$. Then

(1) $T_{a,b}$ is hyponormal for $a, b \in \mathbf{C}$; $|a| \geqq |b|$.

(2) $T^2_{a,b}$ is not hyponormal for $b \neq 0$, $|a| > |b|$.

PROOF. (1) is a well known folk Theorem.

(2) We must show that $A = [T^{*2}_{a,b}, T^2_{a,b}]$ is not positive semi-definite. A straightforward calculation shows that

$$(Ae_2, e_2) = 0 \quad \text{and} \quad (Ae_2, e_0) = \bar{a}b(|a|^2 - |b|^2).$$

Since positivity would imply $0 \neq |(Ae_2, e_0)|^2 \leqq (Ae_2, e_2)(Ae_0, e_0) = 0$ the proof is complete.

THEOREM. *The norm closure of $Q = H_0$.*

PROOF. Clearly the left hand side is contained in the right. To show the reverse inclusion let $\varepsilon > 0$ be given and let $T \in H_0$. If $\lambda \in \sigma_e(T)$, it follows from [2] that $\|T - U^*(\lambda \oplus T)U\| < \varepsilon$ for some unitary operator U. Hence it suffices

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to approximate $(\lambda \oplus T)$, and thus in turn it suffices to approximate λI. Choose $0 < b < a < \varepsilon$. Then

$$\|(\lambda \oplus T) - [(\lambda + aS + bS^*) \oplus T]\| < 2\varepsilon$$

and $(\lambda + aS + bS^* \oplus T) \in Q$ by the Lemma, for $\lambda \neq 0$. For $\lambda = 0$ one can show directly that $aS + bS^* \in Q$ or approximate with $\lambda_n \to 0$.

References

1. P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, NJ, 1967.

2. C. M. Pearcy and N. Salinas, *Compact perturbations of seminormal operators*, Indiana Univ. Math. J. **22** (1973), 789–793.

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